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On shallow water waves in a medium with time-dependent dispersion and nonlinearity coefficients

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ABSTRACT

In this paper, we studied the progression of shallow water waves relevant to the variable coefficient Korteweg–de Vries (vcKdV) equation. We investigated two kinds of cases: when the dispersion and nonlinearity coefficients are proportional, and when they are not linearly dependent. In the first case, it was shown that the progressive waves have some geometric structures as in the case of KdV equation with constant coefficients but the waves travel with time dependent speed. In the second case, the wave structure is maintained when the nonlinearity balances the dispersion. Otherwise, water waves collapse. The objectives of the study are to find a wide class of exact solutions by using the extended unified method and to present a new algorithm for treating the coupled nonlinear PDE's.

Introduction

Many phenomena in physics, biology, chemistry and other fields are described by nonlinear evolution equations (NLEEs). In order to better understand these phenomena, it is important to search for exact solutions to these equations. A variety of methods for obtaining exact solutions of NLEEs have been presented [1–8]. However, to the best of our knowledge, most of the aforementioned methods were related to the constant coefficient models. Recently, a method that unifies all these common methods was suggested by Abdel-Gawad [9]. The study of NLEEs with variable coefficients has attracted much attention, [10–13], because most of real nonlinear physical equations possess variable coefficients.

In this paper, we use the extended unified method which is accomplished by presenting a new algorithm to deal with evolution equations with variable coefficients [14]. This method is an extension to the work done by Abdel-Gawad [9].

For instance, we consider the following (vcKdV) equation

$$H(x,t,u,\ldots) \equiv F(x,t,u,u_x) + f(t)\frac{\partial^m u}{\partial x^m} + \alpha_0 u_t = 0, \tag{1}$$

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where the function F is a polynomial in its arguments, α_0 is a constant.

The traveling wave solutions of (1) satisfy

$$G(u, u', u'', \ldots, u^{(m)}) = 0, \quad u' = \frac{du}{dz}, z = x - ct. \tag{2}$$

Some exact solutions of (1) were found, [15,16], by extrapolating the auto-Bäcklund transformation. The homogeneous balance method was used to find some exact solutions for evolution equations with variable coefficients [17,18].

The extended unified method

In this section, we give a brief description of the extended unified method [9,14].

The extended unified method is characterized by two aspects;

– Constructing the necessary conditions for the existence of solutions of an evolution equation.
– Suggesting a new classification to the different structures of solutions, namely:
  (i) The polynomial function solutions.
  (ii) The rational function solutions.

By the polynomial function solutions, we mean (for example) a polynomial in a function $\phi(x, t)$ that satisfies an auxiliary equation which may be solved to elementary or to special functions. Similar outlines hold in the rational function solutions.

The polynomial function solutions

In this section, we introduce the steps of computations to find the polynomial function solutions for NLEEs by using the extended unified method as they follow:

Step 1: The method asserts that the solution of (1) can be written in the form

$$u(x, t) = \sum_{i=0}^{n} a_i(x, t)\phi^i(x, t), \tag{3}$$

and $\phi(x, t)$ satisfies the auxiliary equations

$$\phi_t^p = \sum_{j=0}^{pk} b_j(x, t)\phi^j, \quad \phi_x^p = \sum_{j=0}^{pk} c_j(x, t)\phi^i, \quad p = 1, 2, \tag{4}$$

together with the compatibility equation

$$\phi_{xt} = \phi_{tx}, \tag{5}$$

where $a_i(x, t)$, $b_j(x, t)$ and $c_j(x, t)$ are arbitrary functions in x and t.

We mention that, the cases when $p = 1$ and $p = 2$ correspond to explicit or implicit elementary solutions and periodic (trigonometric) or elliptic solutions respectively. To determine the relation between n and k, we use the balance condition which is obtained by balancing the highest derivative and the nonlinear term in Eq. (1). The consistency condition determines the values of k such that the polynomial solutions exist.

Step 2: By inserting (3) and (4) into (1), we get a set of equations, namely "the principle equations", which is solved in some of arbitrary functions $a_i(x, t)$, $b_j(x, t)$ and $c_j(x, t)$. The compatibility equation in (5) gives rise to $2k - 1$ equations where $k \geqslant 2$.

Step 3: Solving the auxiliary equations in (4).

Step 4: Evaluating the formal exact solution by using (3).

The variable coefficients KdV equation (vcKdV)

Consider the KdV equation with variable coefficients (vcKdV) [19]

$$v_t + f(t)v_{xxx} + g(t)vv_x = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{6}$$

where $f(t) \neq 0$ and $g(t) \neq 0$ are arbitrary functions. We mention that (6) is well known as a model equation describing the progression of weakly nonlinear and weakly dispersive waves in homogeneous media. Eq. (6) arises in various areas of Mathematical Physics and Nonlinear Dynamics. These include Fluid Dynamics with shallow water waves and Plasma Physics. A particular form of (6) when $f(t) = 1$, $g(t) = \frac{1}{\sqrt{t}}$ and by using the following transformation $v = \sqrt{t}\eta$, Eq. (6) becomes the cylindrical KdV equation or the concentric KdV equation [20]

$$\eta_t + \eta\eta_x + \eta_{xxx} + \frac{1}{2t}\eta = 0. \tag{7}$$

Eq. (7) arises in the study of Plasma Physics. Thus, as a special case the solution of the cylindrical KdV equation will fall out from the solution of (6) that will be obtained, in this paper. Soliton, periodic and Jacobi elliptic function solutions of Eq. (6) have been obtained [10,21], when $f(t) = cg(t)$, where c is a constant.

By using the transformations $x = x$ and $\tau = \int_0^t f(t_1)dt_1$, $t > 0$, Eq. (6) can be written as

$$v_\tau + v_{xxx} + h(\tau)vv_x = 0, \tag{8}$$

where $h(\tau) = \frac{g(\tau)}{f(\tau)} > 0$.

In this work, we use the unified method and the extended unified method to find exact solutions for Eq. (6) when $g(t) = \alpha f(t)$ and $g(t) \neq \alpha f(t)$ respectively, where α is a constant.

When $g(t) = \alpha f(t)$

In this case, Eq. (8) has the traveling wave solution

$$v(x, t) = u(\xi), \quad \xi = ax + b\tau, \tag{9}$$

where a and b are constants. Thus (8) reduces to

$$a^3u''' + a\alpha uu' + bu' = 0, \quad u' = \frac{du}{d\xi}. \tag{10}$$

I – The polynomial function solutions

In this case, we write

$$u(\xi) = \sum_{i=0}^{n} a_i\phi(\xi)^i, \quad (\phi'(\xi))^p = \sum_{j=0}^{pk} c_j\phi(\xi)^j, \quad p = 1, 2. \tag{11}$$

First: when $p = 1$

When $p = 1$, the balance condition yields $n = 2(k-1)$, $k > 1$ and the consistency condition gives rise to $k \leqslant 3$. Thus, in this case, the polynomial function solutions exist when $k = 2, 3$.

(I_1) When $k = 2$, $n = 2$.

By using any package in symbolic computations, we get the solutions of (10) as

$$u(\xi) = -\frac{b + a^3R^2\left(2 + 3tan^2\left(\frac{1}{2}R\xi\right)\right)}{a\alpha}, \tag{12}$$

or

$$u(\xi) = \frac{-b + a^3R_1^2\left(2 - 3tanh^2\left(\frac{1}{2}R_1\xi\right)\right)}{a\alpha}, \quad \xi = ax + b\tau, \tag{13}$$

where $R^2 = 4c_2c_0 - c_1^2 = -R_1^2$ are arbitrary constants. The solution given by (13) is a soliton solution in a moving frame.

Fig. 1a and b represents the solution (13) when $f(t) = 1 + t^2$ in the moving non-inertial frame and in the rest inertial frame respectively.

Fig. 1b shows soliton waves which are moving along the characteristic curve in the xt-plane (namely $ax + b\int_0^t f(t_1)dt_1 = \text{constant}$). The solution in Fig. 1 represents a bright solitary wave solution which is a usual compact solution with a single peak.

(I_2) When $k = 3$, $n = 4$.

By using (11), we have

$$u(\xi) = \sum_{i=0}^{4} a_i\phi(\xi)^i, \quad \phi'(\xi) = \sum_{i=0}^{3} c_i\phi(\xi)^i. \tag{14}$$

By a similar way as we did in the previous case, we get the solution of (10) as

$$u(\xi) = -\frac{b}{a\alpha} - \frac{4a^2(k^2(\xi) - 10k(\xi) + 1)R_0^4}{9c_3^2\alpha(1 + k(\xi))^2},$$
$$k(\xi) = exp\left(-\frac{2R_0^2(27Ac_3^2 + \xi)}{3c_3}\right), \quad \xi = ax + b\tau, \tag{15}$$

where $R_0^2 = c_2^2 - 3c_1c_3$ and A are arbitrary constants.

Second: when $p = 2$

In this case, we find the exact polynomial function solutions for (10) in trigonometric or elliptic functions forms. To this end we put $n = 2$, $k = 1$ or $n = 2$, $k = 2$ in (11) respectively.

(I_1) When $k = 2$, $n = 2$.

By using (11), we have

$$u(\xi) = \sum_{i=0}^{2} a_i\phi(\xi)^i, \quad \phi^{2\prime}(\xi) = c_0 + c_2\phi(\xi)^2 + c_4\phi(\xi)^4. \tag{16}$$

By substituting from (16) into (10) and by using the steps of computations that were given in 'The extended unified method' section, we get

$$a_2 = -\frac{12a^2c_4}{\alpha}, \quad a_1 = 0, \quad a_0 = -\frac{b + 4a^3c_2}{a\alpha}. \tag{17}$$

We mention that c_i, $i = 0, 2, 4$ are arbitrary constants. So the solutions of the auxiliary equation in $(16)_2$ are classified according to Table 1.

In Table 1, $0 < \eta < 1$ is called the modulus of the Jacobi elliptic functions. Detailed recursion equations for the Jacobi elliptic functions can be found (the readers may refer to Refs. [22,23]). When $\eta \to 0$, sn(ξ), cn(ξ) and dn(ξ) degenerate to $sin(\xi)$, $cos(\xi)$ and 1, respectively; while, when $\eta \to 1$, sn(ξ), cn(ξ) and dn(ξ) degenerate to $tanh(\xi)$, sech(ξ) and sech(ξ) respectively.

According to the relation between c_0, c_2 and c_4 in Table 1, we can find the corresponding Jacobi elliptic function solution $\phi(\xi)$.

Finally, the general solution of (10) in terms of the Jacobi elliptic functions is given by

$$u(\xi) = a_2\phi^2(\xi) + a_0, \tag{18}$$

where a_2 and a_0 are given by (17).

Table 1 Relations between the values of (c_0, c_2, c_4) and the corresponding $\phi(\xi)$.

c_4	The relation between (c_0, c_2, c_4)	$\phi(\xi)$
η^2	$c_2 = -(1 + c_4)$, $c_0 = 1$	sn(ξ, η)
$1 - \eta^2$	$c_2 = 1 + c_4$, $c_0 = 1$	sc(ξ, η)
$-\eta^2(1 - \eta^2)$	$c_2^2 = 1 + 4c_4$, $c_0 = 1$	sd(ξ, η)
$\eta^2 - 1$	$c_2 = 1 - c_4$, $c_0 = -1$	nd(ξ, η)
1	$c_2 = -(1 + c_0)$, $c_0 = \eta^2$	ns(ξ, η) = (sn(ξ, η))$^{-1}$
$1 - \eta^2$	$c_2 = 1 - 2c_4$, $c_0 = c_4 - 1$	nc(ξ, η) = (cn(ξ, η))$^{-1}$
-1	$c_2 = 1 - c_0$, $c_0 = \eta^2 - 1$	dn(ξ, η)
$-\eta^2$	$c_2 = -1 - 2c_4$, $c_0 = c_4 + 1$	cn(ξ, η)
1	$c_2 = 1 + c_0$, $c_0 = 1 - \eta^2$	cs(ξ, η)
1	$c_2^2 = 1 + 4c_0$, $c_0 = -\eta^2(1 - \eta^2)$	ds(ξ, η)

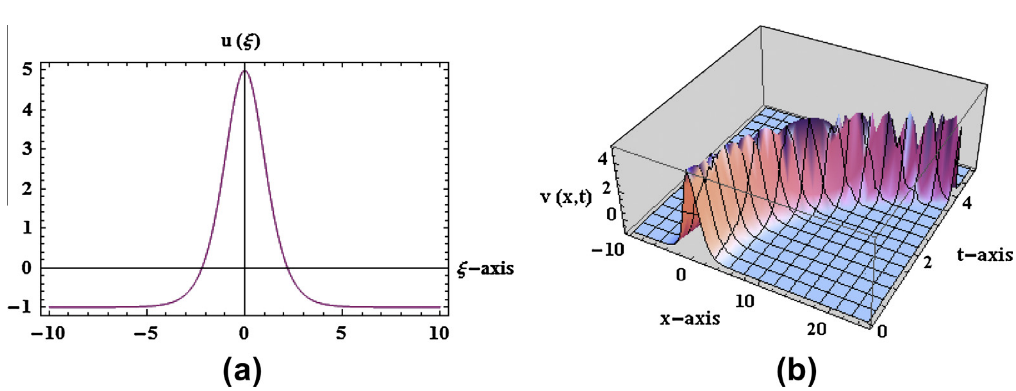

Fig. 1 $\alpha = 1$, $a = 1$, $b = -1$, $R_1 = \sqrt{2}$.

Fig. 2a and b represents the Jacobi doubly periodic solution (18) when $f(t) = 1 + t^2$ and $\phi(\xi) = \mathrm{sn}(\xi, \eta)$, $\xi = ax + bt$ in the moving non-inertial frame and in the rest inertial frame respectively.

II – The rational function solutions

In this section, we find a rational function solution of (10). To this end, we write

$$u(\xi) = \sum_{i=0}^{n} p_i \phi^i(\xi) \Big/ \sum_{j=0}^{r} q_j \phi^j(\xi), \quad n \geqslant r, \tag{19}$$

where p_i and q_j are constants to be determined later, while $\phi(\xi)$ satisfies the previous auxiliary equations in R.H.S. of (11).

In this case, the balance condition is given by $n - r = 2(k-1), k \geqslant 1$ where $n > r$. While k being free when $n = r$.

Here, we confine ourselves to find the rational solutions when $n = r$ and $k = 1, 2$ together with the auxiliary equation in (11) when $p = 2$.

(II_1) When $k = 1$.

In this case, the rational function solutions will be in the rational trigonometric function or hyperbolic function solutions.

– Set $n = r = 1$ (for instance) in (19), namely

$$u(\xi) = \frac{p_1\phi(\xi) + p_0}{q_1\phi(\xi) + q_0}. \tag{20}$$

– Substituting from Eq. (20) together with the auxiliary equation (11) into Eq. (10), we get

$$q_1 = -\frac{ap_1\alpha}{b + a^3c_2},$$
$$q_0 = -\frac{a(-3a^3p_1c_1 + p_0(b + a^3c_2))\alpha}{(b - 5a^3c_2)(b + a^3c_2)},$$
$$p_0 = -\frac{p_1(a^3c_2(c_1 - 5R_2) + b(c_1 + R_2))}{2c_2(b + a^3c_2)}, \tag{21}$$

where $R_2^2 = c_1^2 - 4c_2c_0$ and $c_1^2 \geqslant 4c_2c_0$.

It remains to solve the auxiliary equation in (11). We distinguish between two cases:

Case 1. If $c_2 > 0$. In this case, the solution of the auxiliary equation (11) is

$$\phi(\xi) = -\frac{c_1}{2c_2} + \frac{R_2\cosh(\sqrt{c_2}\xi + A_1)}{2c_2}, \quad \xi = ax + b\tau,$$
$$\tau = \int_0^t f(t_1)dt_1, \tag{22}$$

where A_1 is an arbitrary constant. Substituting (22) into (19) we get the solution of (10), namely

$$u(\xi) = -\frac{b - 5a^3c_2 + (b + a^3c_2)\cosh(\sqrt{c_2}\xi + A_1)}{a\alpha(1 + \cosh(\sqrt{c_2}\xi + A_1))}. \tag{23}$$

Eq. (23) describes a soliton wave solution in the moving frame

Case 2. If $c_2 < 0$. The solution of the auxiliary equation (11) gives

$$\phi(\xi) = -\frac{c_1}{2c_2} + \frac{R_2\sin(\sqrt{-c_2}\xi + A_2)}{2c_2}, \tag{24}$$

where A_2 is an arbitrary constant. Substituting (24) into (19) we get the solution of (10), namely

$$u(\xi) = -\frac{b - 5a^3c_2 + (b + a^3c_2)\sin(\sqrt{-c_2}\xi + A_2)}{a\alpha(1 + \sin(\sqrt{-c_2}\xi + A_2))}. \tag{25}$$

The solutions in (23) and (25) show a soliton wave and a periodic wave solution (as in a rational form) respectively.

(II_2) When $k = 2$.

In this case, the solutions will be in the rational elliptic function form.

To obtain this type of solutions we use the auxiliary equation (11) when $k = 2$. By substituting about $u(\xi)$ from (19) together with $\phi'(\xi)$ from (11) into Eq. (10) and using the calculations that were given in 'The extended unified method' section, we get;

$$p_1 = -\frac{bq_1^2 + a^3(c_2q_1^2 + 6c_4q_0^2)}{aq_1\alpha},$$
$$p_0 = -\frac{bq_0^2 + a^3(6c_0q_1^2 + c_2q_0^2)}{aq_1\alpha},$$
$$q_0 = \sqrt{\frac{R_3 - c_2}{2c_4}}q_1, \tag{26}$$

where $R_3^2 = c_2^2 - 4c_4c_0$, $c_4 > 0$ and $c_0 < 0$. It remains to solve the auxiliary equation in (11). The solutions of the auxiliary

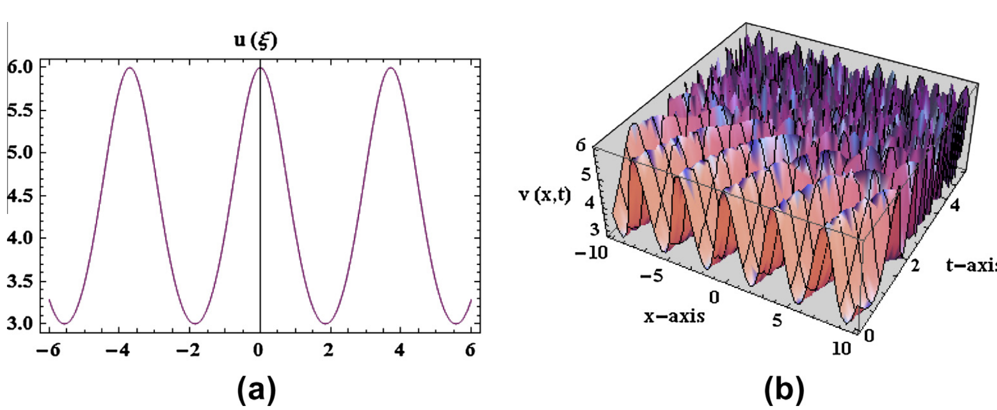

Fig. 2 $\alpha = 1$, $a = 1$, $b = -1$, $c_4 = 0.25$, $c_0 = 0$.

equation in (11) are classified according to Table 1 under the conditions $c_0 < 0$ and $c_4 > 0$.

Finally, the solution of (10) is given by

$$u(\xi) = \frac{\sqrt{2}(-bR_3 + 5a^3c_2R_3 + 6a^3c_4R_3^2 - (b + a^3(c_2 + 6c_4R_3^2))\phi(\xi))}{a\alpha(1 + \sqrt{2}\phi(\xi))}. \tag{27}$$

Fig. 3a and b represents the solution (27) when $f(t) = 1 + t^2$ and $\phi(\xi) = \mathrm{nc}(\xi, \eta)$, $\xi = ax + bt$ in the moving non-inertial frame and in the rest inertial frame respectively.

Fig. 3 shows the propagation of shallow water waves which are seen as elliptic waves.

Indeed, the solutions that were found in the last two cases may cover all solutions which could be obtained by different methods such as a modified tanh–coth method, the Jacobi-elliptic function expansion method, the extended F-expansion method, Exp-function method and $\left(\frac{G'}{G}\right)$-expansion method [24–28].

When $g(t) \neq \alpha f(t)$

In this section, we find exact solutions for Eq. (8) when their coefficients are linearly independent (namely $g(t) \neq \alpha f(t)$). We think that, to the best of our knowledge, the results that will be found here are completely new.

We confine ourselves to search for polynomial function solutions for (8) when $p = 1$ (in (4)) by using (3)–(5). So the balancing condition is $n = 2(k - 1)$, $k > 1$ and the consistency condition for obtaining these polynomial function solutions holds when $k = 2, 3$ [14].

In this case, the calculations are carried out by using the extended unified method together with the symbolic computation for treating coupled nonlinear PDE's according to the following algorithm;

(i) Solve a nonlinear PDE equation among the set of principle or compatibility equation in the highest order (say $\frac{\partial^n w}{\partial x^n}$).
(ii) Solve another equation in $\frac{\partial^{n-1} w}{\partial x^{n-1}}$.
(iii) Use the compatibility equation between (i) and (ii) to eliminate $\left(\frac{\partial^n w}{\partial x^n} \text{ and } \frac{\partial^{n-1} w}{\partial x^{n-1}}\right)$, that is by differentiating the obtained equation in (ii) with respect to x to get $\frac{\partial^n w}{\partial x^n}$ and balances it with the obtained one in (i).
(iv) Solve the obtained equation from (iii) in $\frac{\partial^{n-2} w}{\partial x^{n-2}}$.
(v) Repeat the steps (i)–(iv) to get an equation in the lowest order.
(vi) Use the same steps for PDE's with mixed partial derivatives.

By this algorithm, the order of the PDE is reduced successively till a solution to the required function is obtained.

When $k = 2$, $n = 2$.

The steps of the computations by using the extended unified method (when $p = 1$) are as they follow;

Step 1: Solving the principle equations.

By substituting from (3) and (4) into Eq. (8), we get the principle equation which splits into a set of equations in the unknown functions $a_i(x, \tau)$, $b_i(x, \tau)$ and $c_i(x, \tau)$. For convenience, we use the transformations on $c_i(x, \tau)$ that simplify the computation

$$cc_{2x}(x,\tau) = p(x,\tau)c_2(x,\tau), \quad c_1(x,\tau) = -p(x,\tau) + C_1(x,\tau),$$
$$c_0(x,\tau) = \frac{-2C_{1x}(x,\tau) + C_1^2(x,\tau) + 4C_0(x,\tau)}{4c_2(x,\tau)},$$
$$\tau = \int_0^t f(t_1)dt_1, \tag{28}$$

and we solve the obtained equations to get $b_i(x, \tau)$, $i = 0, 1, 2$, $a_j(x, \tau)$, $j = 1, 2$ and $C_0(x, \tau)$ respectively. We are left with unsolved single equation among them.

Step 2: Solving the compatibility equations in (5).

These equations read

$$\begin{aligned} &b_0(x,\tau)c_1(x,\tau) - b_1(x,\tau)c_0(x,\tau) + c_{0\tau}(x,\tau) - b_{0x}(x,\tau) = 0,\\ &2b_0(x,\tau)c_2(x,\tau) - 2b_2(x,\tau)c_0(x,\tau) + c_{1\tau}(x,\tau) - b_{1x}(x,\tau) = 0,\\ &-b_2(x,\tau)c_1(x,\tau) + b_1(x,\tau)c_2(x,\tau) + c_{2\tau}(x,\tau) - b_{2x}(x,\tau) = 0, \end{aligned} \tag{29}$$

and (28) will be used in (29). Eqs. $(29)_3$ and $(29)_2$ were solved to get $a_{0x}(x, \tau)$ and $a_{0\tau}(x, \tau)$ respectively. The compatibility equation between the obtained results for $a_{0x}(x, \tau)$ and $a_{0\tau}(x, \tau)$ gives rise to an equation which solves to

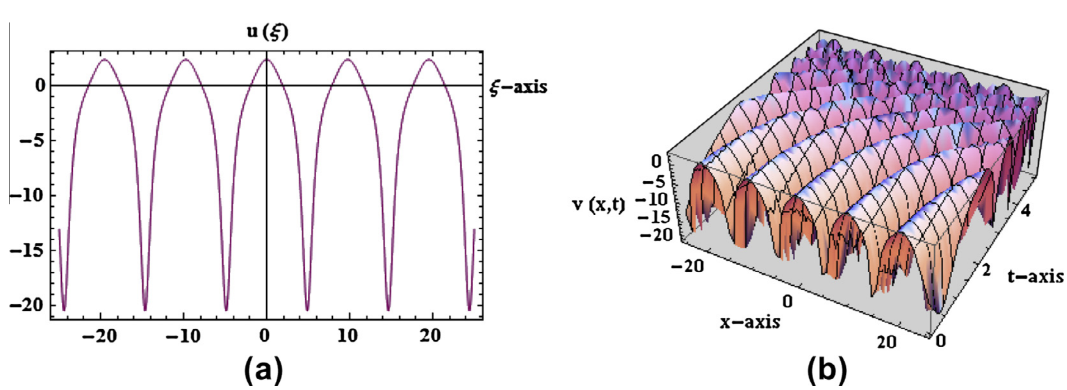

Fig. 3 $\alpha = 1$, $a = 1$, $b = -1$, $c_4 = 0.25$, $c_2 = 0.5$, $c_0 = -0.75$.

$$h(\tau)=\frac{h_1}{\sqrt{h_0+2\tau}} \text{ or}$$
$$f(t)=g(t)\sqrt{k_0+k_1\int_0^t g(t_1)dt_1}, \tag{30}$$

where h_i and k_i, $i = 0, 1$ are arbitrary constants.

By using the obtained result for $a_{0\tau}(x,\tau)$, we found that it satisfies the unsolved equation in the principle ones also. Thus we are only left with Eq. (29)$_1$, which is a nonlinear PDE in $C_0(x,\tau)$, $C_1(x,\tau)$ and $c_2(x,\tau)$. Consequently, we have two arbitrary functions, namely $c_2(x,\tau)$ and $C_1(x,\tau)$, so that no loss of generality if we take $c_2(x,\tau) = 1$ and $C_1(x,\tau) = 0$. Thus (29)$_1$ is closed in $C_0(x,\tau)$. This equation is satisfied by taking $C_0(x,\tau) = A_3h^2(\tau) - \frac{2}{x^2}$ or when $C_0(x,\tau) = A_4h^2(\tau)$, where A_3 and A_4 are constants.

Step 3: Solving the auxiliary equations in (4)$_1$.

In this step Eq. (4)$_2$ is solved in the new variables according to the following two cases;

(i) When $C_0(x,\tau) = A_3h^2(\tau) - \frac{2}{x^2}$

$$\phi_1(x,\tau)=\frac{(h_0+2\tau-h_2x^2)\cos(\mu_1(x,\tau))-\sqrt{h_2(h_0+2\tau)}x\sin(\mu_1(x,\tau))}{\sqrt{h_0+2\tau}x(\sqrt{h_0+2\tau}\cos(\mu_1(x,\tau))-\sqrt{h_2}x\sin(\mu_1(x,\tau)))}, \tag{31}$$

where $\mu_1(x,\tau) = \frac{\sqrt{h_2}(4h_2-x)}{\sqrt{h_0+2\tau}}$, $h_2 > 0$ is a constant or

$$\phi_2(x,\tau)=\frac{(h_0+2\tau+h_3x^2)\cosh(\mu_2(x,\tau))+\sqrt{h_3(h_0+2\tau)}x\sinh(\mu_2(x,\tau))}{\sqrt{h_0+2\tau}x(\sqrt{h_0+2\tau}\cosh(\mu_2(x,\tau))+\sqrt{h_3}x\sinh(\mu_2(x,\tau)))}, \tag{32}$$

where $\mu_2(x,\tau) = \frac{-\sqrt{h_3}(4h_3+x)}{\sqrt{h_0+2\tau}}$, $h_3 > 0$ is a constant.

(ii) When $C_0(x,\tau) = A_4h^2(\tau)$

$$\phi_3(x,\tau)=\frac{A_4h_1^2(\exp(2H(\tau)(2A_4h_1^2+x))-2\sqrt{h_0+2\tau}H(\tau))}{(h_0+2\tau)H(\tau)(exp(2H(\tau)(2A_4h_1^2+x)+2H(\tau)\sqrt{h_0+2\tau})}, \tag{33}$$

where $H(\tau) = \sqrt{-\frac{A_4h_1^2}{h_0+2\tau}}$ and $A_4 < 0$, h_0 are arbitrary constants.

Step 4: Finding the formal solution.

Finally, the solutions of (8) according to the cases (i) and (ii) respectively are given by

$$v_1(x,\tau)=\frac{1}{2s_1\sqrt{h_0+2\tau}(\sqrt{h_0+2\tau}cos(\mu_1(x,\tau))-\sqrt{h_2}x\sin(\mu_1(x,\tau)))^2}$$
$$\times(Q_1(x,\tau)+((h_0+2\tau)x+h_2^2(12h_0+24\tau-x^3))cos(2\mu_1(x,\tau))$$
$$-2\sqrt{h_3(h_0+2\tau)}x^2\sin(2\mu_1(x,\tau))), \tag{34}$$

$$v_2(x,\tau)=\frac{1}{2s_2\sqrt{h_0+2\tau}(\sqrt{h_0+2\tau}\cosh(\mu_2(x,\tau))+\sqrt{h_3}x\sinh(\mu_2(x,\tau)))^2}$$
$$\times(Q_2(x,\tau)+((h_0+2\tau)x+h_3^2(12h_0+24\tau-x^3))cosh(2\mu_2(x,\tau))$$
$$+2\sqrt{h_3(h_0+2\tau)}x^2sinh(2\mu_2(x,\tau))), \tag{35}$$

$$v_3(x,\tau)=\frac{1}{\sqrt{h_0+2\tau}}$$
$$\times\left(x-6A_4h_1^2+\frac{(exp(2H(\tau)(2A_4h_1^2+x))-2H(\tau)\sqrt{h_0+2\tau})^2}{(exp(2H(\tau)(2A_4h_1^2+x))+2H(\tau)\sqrt{h_0+2\tau})^2}\right), \tag{36}$$

where $Q_1(x,\tau) = 12h_0h_2^2 + 24h_2^2\tau + h_0x + 2\tau x - 24h_2^4x^2 + h_2^2x^3$, $Q_2(x,\tau) = 12h_0h_3^2 + 24h_3^2\tau + h_0x + 2\tau x - 24h_3^4x^2 + h_3^2x^3$, $\tau = \int_0^t g(t_1)dt_1$, $t > 0$ and s_i, $i = 1, 2$ are constants.

We mention that the solutions which are given in (34)–(36) satisfy Eq. (8).

Fig. 4a and b represents the solutions in (34) and (35) when $g(t) = 1 + t^2$ respectively.

The solution in Fig. 4a shows the interaction between soliton, solitary and periodic waves (a highly dispersed periodic-soliton waves). While the solution in Fig. 4b shows a soliton wave coupled to two solitary waves the intersection between soliton, kink and anti-kink waves.

II. When $k = 3$, $n = 4$.

By using the same steps in the previous case (when $k = n = 2$), we get the solution of (8) as

$$v(x,\tau)=\frac{1}{h_1Q(\tau)(2Q(\tau)+A_0h_1(s_1+x+s_0Q(\tau)))^2}$$
$$\times((4(s_1+x)(1+A_0h_1s_0)+A_0^2h_1^2(-12+s_0^2(s_1+x)))Q^2(\tau)$$
$$+A_0h_1(s_1+x)^2(4Q(\tau)+A_0h_1(s_1+x+2s_0Q(\tau)))), \tag{37}$$

where $Q(\tau) = \sqrt{h_0+2\tau}$ and s_i, h_i, A_0, $i = 0, 1$ are arbitrary constants. Again, we verified that the solution in (37) satisfies Eq. (8).

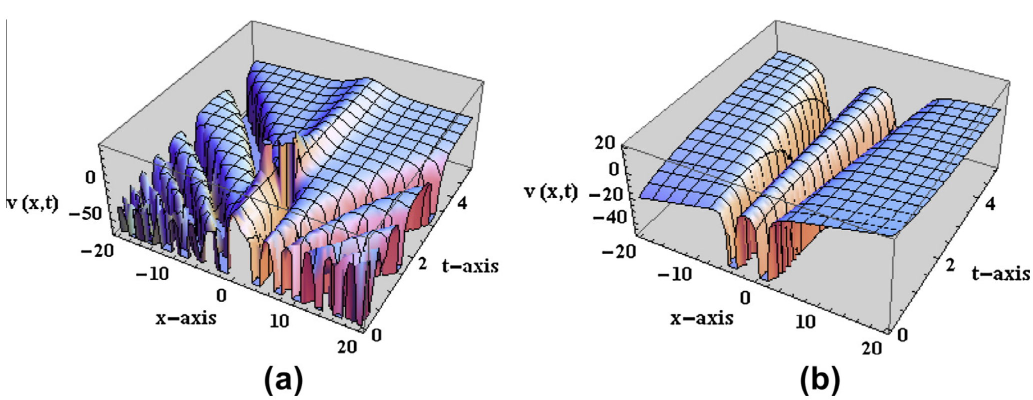

Fig. 4 $\alpha = 1$, $a = 1$, $b = -1$, $c_4 = 0.25$, $c_2 = 0.5$, $c_0 = -0.75$.

Conclusions

The Korteweg–de Vries equation with variable coefficients which describes the shallow water wave propagation through a medium with varying dispersion and nonlinearity coefficients was studied. The extended unified method for finding exact solutions to this equation has been outlined. We have shown that water waves propagate as traveling solitary (or elliptic) waves with anomalous dispersion. This holds when the coefficients of the nonlinear and dispersion terms are linearly dependent (or comparable). For linearly independent coefficients, the water waves behave in similarity waves with a breakdown of wave propagation. This holds when the dispersion coefficients prevail the nonlinearity. Some of these solutions show ‘‘winged’’ soliton (anti-soliton) or wave train solutions. The obtained solutions here are completely new. The extended unified method can be used to find exact solutions of coupled evolution equations, but we think that parallel computations should be used because they require a very lengthy computation. Indeed, they cannot be transformed to traveling wave equations.

Conflict of interest

The authors have declared no conflict of interest.

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

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